

Self-Force on a Scalar Charge in Circular Orbit around a Schwarzschild Black Hole

Hiroyuki NAKANO,^{*)} Yasushi MINO^{*,**)} and Misao SASAKI ^{***)}

*Department of Earth and Space Science, Graduate School of Science
Osaka University, Toyonaka 560-0043, Japan*

**Theoretical Astrophysics, California Institute of Technology
Pasadena, California 91125, USA*

(Received April 12, 2001)

In an accompanying paper, we formulate two types of regularization methods to calculate the scalar self-force on a particle of charge q moving around a black hole of mass M [Y. Mino, H. Nakano and M. Sasaki, *Covariant Self-force Regularization of a Particle Orbiting a Schwarzschild Black Hole*, UTAP-386, OUTAP-157 (2001)], one of which is called the “power expansion regularization”. In this paper, we analytically evaluate the self-force (which we also call the ‘reaction force’) to third post-Newtonian (3PN) order for a scalar particle in circular orbit around a Schwarzschild black hole by using the power expansion regularization. It is found that the r -component of the self-force arises at 3PN order, whereas the t - and ϕ -components, which are due to the radiation reaction, appear at 2PN and 1.5PN orders, respectively.

§1. Introduction

Thanks to recent technological advances, we have almost come to the stage that gravitational waves are detectable. The anticipated observation of gravitational waves represents an absolutely new window to observe our universe. We also expect that the observation of gravitational waves will provide a direct experimental test of general relativity.

There are several on-going projects of gravitational wave detection in the world, such as LIGO,²⁾ VIRGO,³⁾ GEO⁴⁾ and TAMA.⁵⁾ In addition, LISA⁶⁾ is proposed as a future project. LISA is unique in that it is a space-based observation. Since the detector will be launched in space, it will be free from seismic noise, and it will be extremely sensitive to low frequency gravitational waves below 1 Hz. In contrast to the ground-based gravitational wave detectors whose main targets are inspiralling binaries of solar-mass compact objects (either black holes or neutron stars), LISA will detect gravitational waves from solar-mass compact objects orbiting supermassive black holes at galactic centers. In the latter situation, the radiation reaction effect over a few orbital periods will be much smaller than in the case of compact binaries, because of the large mass ratio between the black hole and the compact object. However, the accumulated “secular” effect of the radiation reaction will be as important as in the case of compact binaries. Furthermore, it is expected

^{*)} E-mail: denden@vega.ess.sci.osaka-u.ac.jp

^{**) E-mail: mino@utap.phys.s.u-tokyo.ac.jp (on the leave from University of Tokyo)}

^{***)} E-mail: misao@vega.ess.sci.osaka-u.ac.jp

that the spacetime around a supermassive black hole can be described by the Kerr geometry with spin. Hence, the relativistic effect, and especially the effect of the non-spherical geometry, will significantly affect the orbital evolution. In this situation, the calculation of the orbital evolution by means of the standard post-Newtonian expansion method would seem to be formidable. Instead, the black hole perturbation approach seems much more appropriate, in which a compact object is approximated by a particle orbiting a Kerr black hole (and perturbing the geometry).

For a binary system of almost equal mass objects, the orbit is expected to become sufficiently circular in the final inspiral stage ($\sim 10^5$ cycles before coalescence) due to the radiation reaction. However, this is most probably not the case for binary systems of large mass difference, and the orbit may not become circular until just prior to coalescence. Thus we are required to consider general orbits.

Previous work on the radiation reaction in the black hole perturbation approach has been based on the conservation of the total energy and angular momentum of the system (see Ref. 7) for a review). However, this approach cannot work for general orbits around a Kerr black hole, since such orbits are characterized by the three parameters, the energy, angular momentum (with respect to the symmetry axis), and the Carter constant.⁸⁾ Since the Carter constant is not associated with the Killing vector field of the Kerr geometry, its evolution cannot be determined by the conservation laws. Instead, it is necessary to derive the reaction force acting locally on the orbit.

Recently, the equation of motion of a point particle on a curved background with the effect of gravitational radiation reaction was derived by Mino, Sasaki and Tanaka by solving the Einstein equations by means of a matched asymptotic expansion,⁹⁾ and by Quinn and Wald with an axiomatic approach.¹⁰⁾

The essential problem of the equation of motion of a point particle was that the metric perturbation induced by the particle diverges at the location of the particle. Apparently, divergence of the metric perturbation means the breakdown of the linear approximation. Nevertheless, we may expect that the strong equivalence principle still holds to an extent, and that the concept of the equation of motion of a point particle is still useful. The work of Refs. 9) and 10) has shown that this is indeed the case, and that the extra force due to the particle's self-gravity can be derived from the linear metric perturbation by subtracting the singular part appropriately.

The singular part irrelevant to the motion can be exactly identified by the covariant Hadamard method¹¹⁾ as the coincidence limit of the part that directly propagates to field points along the background null cone of the source. The rest of the metric perturbation is called the 'tail part', which propagates within the null cone due to the curvature scattering. In practice, the direct part can be calculated using the method of DeWitt and Brehme.¹²⁾ On the other hand, the calculation of the tail part requires full knowledge of the metric perturbation, and it is impossible to calculate it for a general spacetime. Fortunately, in the case of the black hole perturbation, it is in principle possible to calculate the full metric perturbation¹³⁾ by using the Regge-Wheeler-Teukolsky formalism.¹⁴⁾ Then we may extract the tail part by subtracting the direct part from the full metric perturbation.

Here, however, we must mention a couple of non-trivial obstacles when we at-

tempt to perform the extraction procedure:

1) Subtraction problem

In the Regge-Wheeler-Teukolsky formalism, the full metric perturbation is calculated in the form of a Fourier-harmonic series.^{*)} This fact totally obscures the local spacetime behavior of the metric perturbation around the particle. Contrastingly, the direct part is given only locally around the location of the particle. This nature of locality makes it very difficult for us to transform the direct part into the Fourier-harmonic series form. Thus the subtraction of the direct part from the full metric perturbation is a highly non-trivial task.

2) Gauge problem

The Hadamard prescription, which identifies the direct part is, by definition, applicable only to hyperbolic differential equations, and specifically to the perturbation equations in the Lorentz (harmonic) gauge in the present case of interest. However, the metric perturbation directly obtainable in the Regge-Wheeler-Teukolsky formalism is not the one in the Lorentz gauge. Hence an appropriate gauge transformation should be worked out.

In addition to the above problems, in the case of a Kerr background, which is our ultimate concern, there is a much more intricate technical issue of how to treat the spheroidal harmonics: Neither the eigenfunction nor the eigenvalue has a simple analytical expression, and they are entangled with the frequency eigenvalues of Fourier modes. Although our ultimate goal is to overcome these difficulties altogether, since each one is sufficiently involved, we choose to proceed step by step, and tackle the subtraction problem first. So we consider the scalar case in this paper, since it is free from the gauge problem.

As a solution to the subtraction problem, we have recently formulated two types of regularization methods,¹⁾ extending earlier works.^{15), 16)} We termed these two methods the ‘power expansion regularization’ and the ‘mode-by-mode regularization’. Although the methods we have developed should be applicable to the gravitational case in principle, we have focused on the scalar case: The Regge-Wheeler or Teukolsky equation reduces to the simple scalar d’Alembertian equation. As for the mode-by-mode regularization, a variant of it adapted to the gravitational case has been applied to a particle in circular orbit around a Schwarzschild black hole in Ref. 17) successfully, but only to 1PN order. The purpose of the present paper is to demonstrate the effectiveness of the power expansion regularization explicitly and analytically by applying it to the simplest case of a scalar particle in circular orbit around a Schwarzschild black hole to 3PN order.

Analytic solutions of the Teukolsky equation were derived by Mano, Suzuki and Takasugi^{18), 19)} in the form of a series of hypergeometric functions. We use their result to construct the Green function to 3PN order, where the technical advantage of the power expansion regularization becomes clear. The direct part has been obtained under the local expansion for general orbits in Ref. 1).

There have appeared some papers investigating the extraction of the scalar self-

^{*)} Throughout the paper, the Fourier modes are considered as $e^{-i\omega t}$ and the harmonic modes as $Y_{\ell m}(\theta, \phi)$.

force.²⁰⁾⁻²⁵⁾ Recently, using the mode-sum regularization prescription proposed in Ref. 20), the self-force acting on an electric or scalar charge has been studied in the spacetime of spherical shells. Recently, the nature of the gravitational self-force has been discussed by Detweiler.²⁶⁾

This paper is organized as follows. In §2, we briefly review the power expansion regularization developed in Ref. 1). In §3, we consider the full Green function to 3PN order in the Fourier-harmonic expanded form and perform the power expansion. In §4, we perform the power expansion of the direct part evaluated with the local expansion. Then, in §5, we derive the regularized scalar self-force by applying the power expansion regularization to the results obtained in the previous two sections. Finally, we summarize our result and discuss possible extensions of our work in §6.

§2. Method of regularization

The reaction force on a point scalar charge is given by

$$F^\alpha(\tau) = \lim_{x \rightarrow z(\tau)} \left(F^\alpha[\phi^{\text{full}}](x) - F^\alpha[\phi^{\text{dir}}](x) \right) = \lim_{x \rightarrow z(\tau)} F^\alpha[\phi^{\text{tail}}](x), \quad (2.1)$$

where ϕ^{full} is the scalar field induced by the point source, ϕ^{dir} is the direct part of the field defined in Refs. 27), 12), 10) and 9), $\phi^{\text{tail}} = \phi^{\text{full}} - \phi^{\text{dir}}$, and $F^\alpha[\cdot \cdot \cdot]$ is a linear tensor derivative operator that derives the force from the scalar field. To obtain ϕ^{full} , we construct the full Green function in the form of a Fourier-harmonic series. Since both the full scalar field and the direct part diverge at the location of the particle, we calculate them at a field point away from the particle, where the subtraction of the divergent part is well-defined. We call this field point x_{reg}^α , the regularization point. There is a degree of freedom in the choice of the regularization point. Let the location of the particle at its proper time $\tau = \tau_0$ be

$$\{z_0^\alpha\} = \{z^\alpha(\tau_0)\} = \{t_0, r_0, \theta_0, \phi_0\}. \quad (2.2)$$

For the power expansion regularization, which will be briefly explained later in this section, we take $\{x_{\text{reg}}^\alpha\} = \{t, r, \theta_0, \phi_0\}$.

We consider a point scalar charge q moving in the Schwarzschild background

$$ds^2 = - \left(1 - \frac{2M}{r} \right) dt^2 + \left(1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.3)$$

where $\{x^\alpha\} = \{t, r, \theta, \phi\}$ are the Schwarzschild coordinates, and M is the black hole mass. The charge density of the scalar particle in circular orbit is given by

$$\rho(x^\alpha) = q \int d\tau \frac{\delta^{(4)}(x^\alpha - z^\alpha(\tau))}{\sqrt{-g}}, \quad (2.4)$$

where g denotes the determinant of the Schwarzschild metric, z^α is the trajectory of the particle,

$$z^\alpha(\tau) = \left\{ v^t \tau, r_0, \frac{\pi}{2}, v^\phi \tau \right\}; \quad v^t = \sqrt{\frac{r_0}{r_0 - 3M}}, \quad v^\phi = \frac{1}{r_0} \sqrt{\frac{M}{r_0 - 3M}}, \quad (2.5)$$

and v^α is the four-velocity of the particle. We have assumed the orbit is on the equatorial plane without loss of generality.

The full scalar field is calculated by using the Green function method as

$$\phi^{\text{full}}(x) = q \int d\tau G^{\text{full}}(x, z(\tau)), \quad (2.6)$$

where the Green function satisfies the wave equation

$$\nabla^\alpha \nabla_\alpha G^{\text{full}}(x, x') = \frac{\delta^{(4)}(x - x')}{\sqrt{-g}}, \quad (2.7)$$

and we impose retarded boundary conditions on $G^{\text{full}}(x, x')$. The full Green function is decomposed into the Fourier-harmonic modes as

$$G^{\text{full}}(x, x') = \int \frac{d\omega}{2\pi} \frac{\omega}{|\omega|} e^{-i\omega(t-t')} \sum_{\ell m} g_{\ell m \omega}^{\text{full}}(r, r') Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi'). \quad (2.8)$$

Then the wave equation (2.7) reduces to an ordinary differential equation for the radial Green function as

$$\left[\left(1 - \frac{2M}{r}\right) \frac{d^2}{dr^2} + \frac{2(r-M)}{r^2} \frac{d}{dr} + \left(\frac{\omega^2}{1 - \frac{2M}{r}} - \frac{\ell(\ell+1)}{r^2} \right) \right] g_{\ell m \omega}^{\text{full}}(r, r') = \frac{1}{r^2} \delta(r - r'). \quad (2.9)$$

We construct the radial function of the full Green function by using homogeneous solutions of (2.9), which can be obtained using a systematic analytic method developed in Ref. 18).

The local expansion of the full Green function gives the direct part of the scalar field,

$$\phi^{\text{dir}}(x) = q \int d\tau G^{\text{dir}}(x, z(\tau)), \quad (2.10)$$

where G^{dir} is the direct part of the Green function. It is given in a covariant manner as

$$G^{\text{dir}}(x, x') = -\frac{1}{4\pi} \theta[\Sigma(x), x'] \sqrt{\Delta(x, x')} \delta(\sigma(x, x')), \quad (2.11)$$

where $\sigma(x, x')$ is the bi-scalar representing half the squared geodesic distance, $\Delta(x, x')$ is the generalized van Vleck-Morette determinant bi-scalar, $\Sigma(x)$ is an arbitrary spacelike hypersurface containing x , and $\theta[\Sigma(x), x'] = 1 - \theta[x', \Sigma(x)]$ is equal to 1 when x' lies in the past of $\Sigma(x)$ and vanishes when x' lie in the future.

We now briefly describe the power expansion regularization procedure. As mentioned above, for a given point on the particle trajectory (2.2), we choose the regularization point as

$$\{x_{\text{reg}}^\alpha\} = \{t, r, \theta_0, \phi_0\}, \quad (2.12)$$

which is assumed to be on a suitably chosen spacelike hypersurface (see Eq. (3.10) in Ref. 1)). It should be noted that t is a function of r . In the following, we set $\tau_0 = 0$ for simplicity.

Assuming $(r - r_0)/r_0 \ll 1$, we expand the full scalar field and its direct part in powers of the radius r as

$$\phi^{\text{full}}(x_{\text{reg}}) = \sum_n r^n \phi^{\text{full}(n)}, \quad \phi^{\text{dir}}(x_{\text{reg}}) = \sum_n r^n \phi^{\text{dir}(n)}. \quad (2.13)$$

Similarly, for the calculation of the self-force, we expand $\nabla_\mu \phi^{\text{full}}$ and $\nabla_\mu \phi^{\text{dir}}$ as

$$\nabla_\mu \phi^{\text{full}}(x_{\text{reg}}) = \sum_n r^n \phi_\mu^{\text{full}(n)}, \quad \nabla_\mu \phi^{\text{dir}}(x_{\text{reg}}) = \sum_n r^n \phi_\mu^{\text{dir}(n)}. \quad (2.14)$$

For ϕ^{full} and $\nabla_\mu \phi^{\text{full}}$, this expansion is done first for each spherical harmonic mode, followed by a summation of the Fourier-harmonic components, to obtain $\phi^{\text{full}(n)}$ and $\phi_\mu^{\text{full}(n)}$. The heart of the power expansion regularization resides in the fact that the coefficients $\phi^{\text{full}(n)}$ and $\phi_\mu^{\text{full}(n)}$ as well as $\phi^{\text{dir}(n)}$ and $\phi_\mu^{\text{dir}(n)}$ of the power series are finite, though these series as a whole diverge in the limit $r \rightarrow r_0$. Hence we may extract the tail part as

$$\begin{aligned} \phi^{\text{tail}}(x_{\text{reg}}) &= \sum_n r^n \phi^{\text{tail}(n)}; & \phi^{\text{tail}(n)} &= \phi^{\text{full}(n)} - \phi^{\text{dir}(n)}, \\ \nabla_\mu \phi^{\text{tail}}(x_{\text{reg}}) &= \sum_n r^n \phi_\mu^{\text{tail}(n)}; & \phi_\mu^{\text{tail}(n)} &= \phi_\mu^{\text{full}(n)} - \phi_\mu^{\text{dir}(n)}. \end{aligned} \quad (2.15)$$

The tail part of the scalar field is obtained by summing over n and taking the limit $r \rightarrow r_0$. This extraction procedure is called the ‘power expansion regularization’.

§3. Power expansion of the full scalar field

We first consider the full Green function decomposed into the Fourier-harmonic modes (2.8). The retarded radial Green function becomes

$$\begin{aligned} g_{\ell m \omega}^{\text{full}}(r, r') &= \frac{1}{W_{\ell m \omega}(\phi^{\text{in}}, \phi^{\text{up}})} \left(\phi_{\ell m \omega}^{\text{in}}(r) \phi_{\ell m \omega}^{\text{up}}(r') \theta(r' - r) + \phi_{\ell m \omega}^{\text{up}}(r) \phi_{\ell m \omega}^{\text{in}}(r') \theta(r - r') \right); \\ W_{\ell m \omega}(\phi^{\text{in}}, \phi^{\text{up}}) &= r^2 \left(1 - \frac{2M}{r} \right) \left(\left(\frac{d}{dr} \phi_{\ell m \omega}^{\text{in}}(r) \right) \phi_{\ell m \omega}^{\text{up}}(r) - \left(\frac{d}{dr} \phi_{\ell m \omega}^{\text{up}}(r) \right) \phi_{\ell m \omega}^{\text{in}}(r) \right), \end{aligned} \quad (3.1)$$

where $\phi_{\ell m \omega}^{\text{in}}$ is a homogeneous solution that vanishes on the past horizon (when multiplied by $e^{-i\omega t}$), and $\phi_{\ell m \omega}^{\text{up}}$ is a homogeneous solution that vanishes at the past null infinity.²⁸⁾ They are called the in-going and up-going solutions, respectively.

Homogeneous solutions of (2.9) were fully investigated in Refs. 18) and 19), and we have various analytic expressions for the homogeneous functions. In this paper, we consider the slow-motion expansion, in which we assume

$$\omega r \approx v, \quad \omega M \approx v^3; \quad v \ll 1. \quad (3.2)$$

Detailed properties of the radial Green function for the calculation up to 3PN order are described in Appendix A. For technical reasons, we classify the harmonic components of the Green function into the three parts:

$$G_{(\ell \geq 2)}^{(\text{sym})}(x, x') = \int \frac{d\omega}{2\pi} \frac{\omega}{|\omega|} e^{-i\omega(t-t')} \sum_{\ell \geq 2, m} g_{\ell m \omega}^{(\text{sym})}(r, r') Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi'), \quad (3.3)$$

$$G_{(\ell \geq 2)}^{(\text{rad})}(x, x') = \int \frac{d\omega}{2\pi} \frac{\omega}{|\omega|} e^{-i\omega(t-t')} \sum_{\ell \geq 2, m} g_{\ell m \omega}^{(\text{rad})}(r, r') Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi'), \quad (3.4)$$

$$G_{(\ell=0,1)}^{\text{full}}(x, x') = \int \frac{d\omega}{2\pi} \frac{\omega}{|\omega|} e^{-i\omega(t-t')} \sum_{\ell=0,1,m} g_{\ell m \omega}^{\text{full}}(r, r') Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi'). \quad (3.5)$$

The suffices (sym) and (rad) indicate that these are the time-symmetric and radiative parts, respectively, of the Green function. That is, we write the (retarded) Green function as

$$G^{\text{full}} = G^{(\text{sym})} + G^{(\text{rad})};$$

$$G^{(\text{sym})} = \frac{1}{2}(G^{(\text{ret})} + G^{(\text{adv})}), \quad G^{(\text{rad})} = \frac{1}{2}(G^{(\text{ret})} - G^{(\text{adv})}), \quad (3.6)$$

where $G^{(\text{adv})}$ is the advanced Green function. Correspondingly, the full scalar field is divided into three parts as

$$\phi_{(\ell \geq 2)}^{(\text{sym})}(x) = \int d^4 x' \sqrt{-g(x')} G_{(\ell \geq 2)}^{(\text{sym})}(x, x') \rho(x'), \quad (3.7)$$

$$\phi_{(\ell \geq 2)}^{(\text{rad})}(x) = \int d^4 x' \sqrt{-g(x')} G_{(\ell \geq 2)}^{(\text{rad})}(x, x') \rho(x'), \quad (3.8)$$

$$\phi_{(\ell=0,1)}^{\text{full}}(x) = \int d^4 x' \sqrt{-g(x')} G_{(\ell=0,1)}^{\text{full}}(x, x') \rho(x'). \quad (3.9)$$

The reason for the division of the Green function into harmonic modes with $\ell \geq 2$ and $\ell = 0, 1$ is purely technical, and is needed because of some non-systematic analytical behavior of the modes $\ell = 0, 1$, as discussed in Appendix A. On the other hand, there is a physical reason for dividing the Green function into the time-symmetric part and the radiative part. It is known that the radiation reaction to the energy and angular momentum of the particle (i.e., any conserved quantity associated with a background Killing vector field) can be calculated by using the radiative Green function $G^{(\text{rad})}$.²⁹⁾ In our case of circular orbit around a Schwarzschild black hole, we therefore expect no contribution from the time-symmetric part to the t - and ϕ -components of the reaction force. Nevertheless, one cannot simply discard the time-symmetric part, since the tail part is not identical to the radiative part. In particular, the r -component of the reaction force, which does not contribute to the

energy and angular momentum loss in our case, cannot be obtained from the radiative part. Since our purpose here is to demonstrate the validity of the power expansion regularization, we regularize $G_{(\ell \geq 2)}^{(\text{sym})}$ to demonstrate the vanishing of the t - and ϕ -components of the reaction force due to the time-symmetric part explicitly, as well as to derive the non-vanishing r -component of the reaction force. The Green functions $G_{(\ell \geq 2)}^{(\text{rad})}$ and $G_{(\ell=0,1)}^{\text{full}}$ are always finite. Hence the power expansion is unnecessary for these parts.

The time-symmetric part of the radial Green function is expanded in powers of r as

$$g_{\ell m \omega}^{(\text{sym})}(r, r') = \sum_n \left(\theta(r' - r) \frac{r^{n+\ell}}{r'^{n+\ell+1}} g_{\ell m \omega}^{\text{in}(n)}(r') + \theta(r - r') \frac{r'^{n+\ell}}{r^{n+\ell+1}} g_{\ell m \omega}^{\text{out}(n)}(r') \right). \quad (3.10)$$

We note that because of the singularity at $r = r_0$, we have two different forms of the power expansion for $0 < r < r_0$ and $r_0 < r$. The final regularized force, however, is the same for either choice of the regularization point.

For the calculation to 3PN order, it is found that we only need $g_{\ell m \omega}^{\text{in}(n)}$ for $-3 \leq n \leq 6$ and $g_{\ell m \omega}^{\text{out}(n)}$ for $-6 \leq n \leq 3$. The coefficients $g_{\ell m \omega}^{\text{in}(n)}$ and $g_{\ell m \omega}^{\text{out}(n)}$ are listed in Appendix B. After summing over ω , ℓ and m , we obtain the power expansion of the time-symmetric Green function as

$$G_{(\ell \geq 2)}^{(\text{sym})}(x, x') = \int \frac{d\omega}{2\pi} \frac{\omega}{|\omega|} e^{-i\omega(t-t')} \sum_n \left(\theta(r' - r) \frac{r^n}{r'^{n+1}} G_{(\ell \geq 2)}^{\text{in}(n)}(\theta, \phi; r', \theta', \phi'; \omega) + \theta(r - r') \frac{r'^n}{r^{n+1}} G_{(\ell \geq 2)}^{\text{out}(n)}(\theta, \phi; r', \theta', \phi'; \omega) \right), \quad (3.11)$$

where

$$G_{(\ell \geq 2)}^{\text{in/out}(n)}(\theta, \phi; r', \theta', \phi'; \omega) = \sum_{\ell \geq 2, m} g_{\ell m \omega}^{\text{in/out}(n-\ell)}(r') Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi'). \quad (3.12)$$

One sees that only a finite number of harmonic modes contribute to each expansion coefficient $G^{\text{in/out}(n)}$ because of the truncation of the terms with large $|n - \ell|$. This greatly simplifies the calculation.

Now, the time-symmetric part of the scalar field (3.7) becomes

$$\phi_{(\ell \geq 2)}^{(\text{sym})}(x) = \frac{q}{v^t} \sum_n \left(\theta(r_0 - r) \frac{r^n}{r_0^{n+1}} \phi^{\text{in}(n)}(r_0) + \theta(r - r_0) \frac{r_0^n}{r^{n+1}} \phi^{\text{out}(n)}(r_0) \right), \quad (3.13)$$

where

$$\phi^{\text{in/out}(n)} = \sum_{\ell m} g_{\ell m, m\Omega}^{\text{in/out}(n-\ell)}(r_0) Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\pi/2, 0) e^{-im\Omega t}, \quad (3.14)$$

and we have introduced the angular frequency $\Omega = v^\phi/v^t$. Taking the derivative of the scalar field, we obtain the power expanded scalar field at the regularization point as

$$\nabla_t \phi_{(\ell \geq 2)}^{(\text{sym})}(x_{\text{reg}}) = 0, \quad (3.15)$$

$$\nabla_r \phi_{(\ell \geq 2)}^{(\text{sym})}(x_{\text{reg}}) = \frac{q}{v^t} \sum_n \left(\theta(r_0 - r) n \frac{r^{n-1}}{r_0^{n+1}} \phi_{(r)}^{\text{in}(n)}(r_0) - \theta(r - r_0)(n+1) \frac{r_0^n}{r^{n+2}} \phi_{(r)}^{\text{out}(n)}(r_0) \right), \quad (3.16)$$

$$\nabla_\theta \phi_{(\ell \geq 2)}^{(\text{sym})}(x_{\text{reg}}) = 0, \quad (3.17)$$

$$\nabla_\phi \phi_{(\ell \geq 2)}^{(\text{sym})}(x_{\text{reg}}) = -\frac{1}{\Omega} \nabla_t \phi_{(\ell \geq 2)}^{(\text{sym})}(x_{\text{reg}}) = 0, \quad (3.18)$$

where

$$\phi_{(r)}^{\text{in/out}(n)} = \sum_{\ell m} g_{\ell m, m\Omega}^{\text{in/out}(n-\ell)}(r_0) |Y_{\ell m}(\pi/2, 0)|^2. \quad (3.19)$$

The fact that the θ -component vanishes results from our assumption of circular orbit, while the reason for the vanishing t - and ϕ -components is somewhat subtle. It results from both the assumption of circular orbit and the symmetry of $g_{\ell m \omega}^{(\text{sym})}$ under $\omega \leftrightarrow -\omega$. The formula used for the m -sum in Eq. (3.19) is presented in Appendix C.

§4. Power expansion of the direct part

Here we evaluate the derivative of the direct part of the scalar field under the local expansion. The details of this evaluation procedure are discussed in Ref. 1). By calculating Eqs. (3.23)–(3.25) in Ref. 1), we obtain the regularized direct part of the scalar reaction force as

$$\nabla_t \phi^{\text{dir}}(x_{\text{reg}}) = O(r - r_0), \quad (4.1)$$

$$\begin{aligned} \nabla_r \phi^{\text{dir}}(x_{\text{reg}}) &= \frac{q}{4\pi} \sqrt{1 - \frac{2M}{r_0} \frac{\text{sgn}(r - r_0)}{(r - r_0)^2}} \\ &\quad \times \left(1 - \frac{1}{4} \frac{M^2(r_0 - M)}{r_0^2(r_0 - 2M)^2(r_0 - 3M)} (r - r_0)^2 \right) \\ &\quad + O(r - r_0), \end{aligned} \quad (4.2)$$

$$\nabla_\theta \phi^{\text{dir}}(x_{\text{reg}}) = O(r - r_0), \quad (4.3)$$

$$\nabla_\phi \phi^{\text{dir}}(x_{\text{reg}}) = O(r - r_0), \quad (4.4)$$

where $\text{sgn}(r - r_0) = \pm 1$ for $r - r_0 \gtrless 0$, and we have ignored terms of order $r - r_0$, as they do not contribute to the force in the coincidence limit $r \rightarrow r_0$. It should be mentioned that the above are exact in the sense that no slow-motion approximation is employed. Note that the second term proportional to M^2/r_0^2 in the parentheses

on the right-hand side of the radial component gives a manifestly finite contribution in the limit $r \rightarrow r_0$. Of course, these simple forms of the direct force components are due to the assumption of circular orbit: They would be more complicated if the radial velocity is non-zero.

Using the formula

$$\frac{1}{(r - r_0)^2} = \sum_{n \geq 0} \left(\theta(r - r_0)(n + 1) \frac{r_0^n}{r^{n+2}} + \theta(r_0 - r)(n + 1) \frac{r^n}{r_0^{n+2}} \right), \quad (4.5)$$

we finally obtain the power expansion of the direct part as

$$\nabla_t \phi^{\text{dir}}(x_{\text{reg}}) = 0, \quad (4.6)$$

$$\begin{aligned} \nabla_r \phi^{\text{dir}}(x_{\text{reg}}) &= \frac{q}{4\pi} \sqrt{1 - \frac{2M}{r_0}} \\ &\times \left\{ \theta(r - r_0) \left(\sum_{n \geq 0} (n + 1) \frac{r_0^n}{r^{n+2}} - \frac{1}{4} \frac{M^2(r_0 - M)}{r_0^2(r_0 - 2M)^2(r_0 - 3M)} \right) \right. \\ &\quad \left. - \theta(r_0 - r) \left(\sum_{n \geq 0} (n + 1) \frac{r^n}{r_0^{n+2}} - \frac{1}{4} \frac{M^2(r_0 - M)}{r_0^2(r_0 - 2M)^2(r_0 - 3M)} \right) \right\}, \quad (4.7) \end{aligned}$$

$$\nabla_\theta \phi^{\text{dir}}(x_{\text{reg}}) = 0, \quad (4.8)$$

$$\nabla_\phi \phi^{\text{dir}}(x_{\text{reg}}) = 0. \quad (4.9)$$

We find that only the r -component of the force diverges in the limit $r \rightarrow r_0$.

§5. Extraction of the scalar reaction force

Now, we are ready to investigate the extraction of the tail part under the power expansion regularization. Since the r -component of the full (bare) force is the only one that diverges in the coincidence limit, we first consider the simpler cases of the t -, θ - and ϕ -components, and separately treat the r -component afterwards.

For the θ -component, all the parts of the force vanish identically, because of the assumption of circular orbit,

$$\nabla_\theta \phi^{\text{tail}}(z) = 0. \quad (5.1)$$

For the t - and ϕ -components, which are equal modulo the factor $-1/\Omega$, both the time-symmetric part and the direct part of the force vanish identically, as given in Eqs. (3.15) and (4.9). Hence the only possible contributions come from $\nabla_t \phi_{(\ell \geq 2)}^{(\text{rad})}$ and $\nabla_t \phi_{(\ell=0,1)}$, in accordance with our expectation. We find the former contribution arises at 3PN order and the latter at 2PN order, which are added together to give

$$\begin{aligned} \nabla_t \phi^{\text{tail}}(z) &= \nabla_t \phi_{(\ell \geq 2)}^{(\text{rad})}(z) + \nabla_t \phi_{(\ell=0,1)}(z) \\ &= \frac{q}{4\pi r_0^2} \left(\frac{1}{3} v^4 + \frac{46073}{30} v^6 + O(v^7) \right), \quad (5.2) \end{aligned}$$

$$\begin{aligned}\nabla_\phi \phi^{\text{tail}}(z) &= -\frac{1}{\Omega} \nabla_t \phi^{\text{tail}}(z) \\ &= -\frac{q}{4\pi r_0} \left(\frac{1}{3} v^3 + \frac{46073}{30} v^5 + O(v^7) \right).\end{aligned}\quad (5.3)$$

It is easy to check that the leading terms of the above are consistent with those of the energy loss rate through scalar radiation evaluated from the energy-momentum tensor of the scalar field at future null-infinity. The leading terms are also in agreement with those derived by Gal'tsov.²⁹⁾

Now we turn to the r -component of the reaction force. Since only the subtraction of the divergent terms is non-trivial, taking account of the form of $\nabla_r \phi^{\text{dir}}(x_{\text{reg}})$ given by Eq. (4.2), we introduce the auxiliary coefficients $f^{\text{in/out}(n)}$ in the extraction procedure of the tail part from $\nabla_r \phi_{(\ell \geq 2)}^{(\text{sym})}(x_{\text{reg}})$ as

$$\begin{aligned}\nabla_r \phi_{(\ell \geq 2)}^{(\text{sym})}(x_{\text{reg}}) &- \frac{q}{4\pi} \sqrt{1 - \frac{2M}{r_0} \frac{\text{sgn}(r - r_0)}{(r - r_0)^2}} \\ &= \frac{q}{4\pi} \sum_n \left(\theta(r_0 - r) n \frac{r^{n-1}}{r_0^{n+1}} f^{\text{in}(n)} - \theta(r - r_0) (n+1) \frac{r_0^n}{r^{n+2}} f^{\text{out}(n)} \right).\end{aligned}\quad (5.4)$$

Then we obtain

$$f^{\text{in}(n)} = O(v^7), \quad (n < 1) \quad (5.5)$$

$$f^{\text{in}(1)} = 1 + v^2 + \frac{11}{10} v^4 + \frac{331}{100} v^6 + O(v^7), \quad (5.6)$$

$$f^{\text{in}(2)} = -\frac{33}{20} v^4 - \frac{173713}{14700} v^6 + O(v^7), \quad (5.7)$$

$$f^{\text{in}(3)} = -\frac{3}{10} v^2 - \frac{3}{10} v^4 + \frac{215419}{22050} v^6 + O(v^7), \quad (5.8)$$

$$f^{\text{in}(4)} = -\frac{30154441}{9604980} v^6 + O(v^7), \quad (5.9)$$

$$f^{\text{in}(5)} = \frac{1}{56} v^4 + \frac{237035}{3091888} v^6 + O(v^7), \quad (5.10)$$

$$f^{\text{in}(6)} = -\frac{44372}{10735725} v^6 + O(v^7), \quad (5.11)$$

$$f^{\text{in}(7)} = -\frac{1288333}{527482800} v^6 + O(v^7), \quad (5.12)$$

$$\begin{aligned}f^{\text{in}(n)} &= \frac{-6n(224n^4 - 224n^3 - 1972n^2 - 912n + 939)}{(2n-7)(2n-5)(2n-3)(2n-1)^2(2n+1)^2(2n+3)^2} v^6 + O(v^7), \\ &\quad (n > 7)\end{aligned}\quad (5.13)$$

$$f^{\text{out}(n)} = O(v^7), \quad (n < -4) \quad (5.14)$$

$$f^{\text{out}(-4)} = v^6 + O(v^7), \quad (5.15)$$

$$f^{\text{out}(-3)} = -\frac{19}{15} v^6 + O(v^7), \quad (5.16)$$

$$f^{\text{out}(-2)} = -\frac{1}{2} v^4 + \frac{13}{9} v^6 + O(v^7), \quad (5.17)$$

$$f^{\text{out}(-1)} = O(v^7), \quad (5.18)$$

$$f^{\text{out}(0)} = -1 + \frac{3}{2}v^2 - \frac{7}{8}v^4 - \frac{7079}{5040}v^6 + O(v^7), \quad (5.19)$$

$$f^{\text{out}(1)} = -2 + \frac{16}{5}v^2 + \frac{117}{35}v^4 - \frac{58711}{69300}v^6 + O(v^7), \quad (5.20)$$

$$f^{\text{out}(2)} = -6v^2 + \frac{58}{5}v^4 + \frac{3686944}{525525}v^6 + O(v^7), \quad (5.21)$$

$$f^{\text{out}(3)} = -\frac{72}{5}v^4 + \frac{12656296}{429975}v^6 + O(v^7), \quad (5.22)$$

$$f^{\text{out}(4)} = -\frac{3396569326}{106135029}v^6 + O(v^7), \quad (5.23)$$

$$f^{\text{out}(n)} = \frac{-6(n+1)(224n^4 + 1120n^3 + 44n^2 - 1464n + 327)}{(2n-1)^2(2n+1)^2(2n+3)^2(2n+5)(2n+7)(2n+9)}v^6 + O(v^7), \quad (n > 4) \quad (5.24)$$

where $v = \Omega r_0$ (and $\Omega M = v^3$).

As for the parts $\nabla_r \phi_{(\ell \geq 2)}^{(\text{rad})}$ and $\nabla_r \phi_{(\ell=1,0)}$, we find the former is higher than 3PN order, and only the latter contributes at 3PN order. In fact, the radiative Green function does not contribute to the radial force for a circular orbit to all orders (see Appendix A).

By summing up (5.4) and adding the finite terms in the direct part together with $\nabla_r \phi_{(\ell=1,0)}$, we finally find

$$\begin{aligned} \nabla_r \phi^{\text{tail}}(z) &= \lim_{r \rightarrow r_0 \pm 0} \left(\nabla_r \phi^{(\text{sym})}(x_{\text{reg}}) - \nabla_r \phi^{\text{dir}}(x_{\text{reg}}) \right) + \nabla_r \phi_{(\ell \geq 2)}^{(\text{rad})}(z) + \nabla_r \phi_{(\ell=0,1)}(z) \\ &= -\frac{q}{4\pi r_0^2} \left(\frac{31}{12}v^6 - \frac{7}{64}\pi^2 v^6 + \frac{8}{3}v^6 \ln(v) + O(v^7) \right). \end{aligned} \quad (5.25)$$

As expected, either choice of the regularization point, at $r < r_0$ or $r > r_0$, gives the same reaction force in the limit $r \rightarrow r_0$. It is noted that the above result is consistent with that of the numerical calculation reported in Ref. 21).

§6. Conclusion

In a separate paper,¹⁾ we propose two technical methods to derive the scalar reaction force on a particle in the Schwarzschild background, the power expansion regularization and the mode-by-mode regularization. We expect these methods will be useful also for the case of a gravitational reaction force. In fact, a variant of the mode-by-mode regularization method has been successfully applied to the gravitational case, although only the 1PN calculation for circular orbit has been performed.¹⁷⁾

In this paper, we have demonstrated the usefulness of the other regularization method, the power expansion regularization method, by applying it to a scalar particle in circular orbit around a Schwarzschild black hole and calculating the scalar reaction force to 3PN order. In the power expansion regularization method, the tail part of the scalar field, which is responsible for the reaction force, is obtained by

subtracting the direct part from the full scalar field at a regularization point specified by the radial coordinate r , followed by taking the limit $r \rightarrow r_0$, where r_0 is the radial coordinate of the particle. We have found that the r -component of the scalar reaction force arises at 3PN order, and the t - and ϕ -components at 2PN and 1.5PN orders, respectively.

Since the derivation of the reaction force for non-circular orbits is very important for binary systems of large mass difference, and we expect that the gravitational waves from such binaries will be detected by future space-based gravitational wave detectors, such as LISA, it is indispensable for us to proceed to the case of general orbits. In this respect, although the calculation necessarily becomes more complicated when we consider non-circular orbits when there exists radial motion, the power expansion regularization method seems to still be applicable. A general framework for treating general orbits in terms of the power expansion regularization scheme has been developed in Ref. 1). As a next step, we plan to consider eccentric and radial orbits with these regularization methods.

Acknowledgements

We thank U. Gen, Y. Himemoto and F. Takahara for fruitful conversations. Special thanks are due to H. Tagoshi and T. Tanaka for valuable discussions. HN and YM also thank K. S. Thorne for his hospitality and discussions during their stay at the California Institute of Technology, and YM thanks B. Schutz for his hospitality and discussions during his stay at the Einstein Institute in Potsdam. This work was supported in part by a Grant-in-Aid for Creative Research, No. 09NP0801, and by a Grant-in-Aid for Scientific Research, No. 12640269, from the Ministry of Education, Science, Sports and Culture. HN and YM are supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists, Nos. 2397 and 0704.

Appendix A

— Full Green Function —

In this appendix, we summarize the derivation of the full radial Green function. Here the in-going and up-going homogeneous solutions are denoted by ϕ_{in}^ν and ϕ_{up}^ν , respectively, where ν is called the ‘renormalized angular momentum’,^{18), 19)} and is equal to ℓ in the limit $\omega M \rightarrow 0$.

The radial Green function is expressed as

$$\begin{aligned}
 g_{\ell m \omega}^{\text{full}}(r, r') &= \frac{1}{W_{\ell m \omega}(\phi_{\text{in}}^\nu, \phi_{\text{up}}^\nu)} \left(\phi_{\text{in}}^\nu(r) \phi_{\text{up}}^\nu(r') \theta(r' - r) + \phi_{\text{up}}^\nu(r) \phi_{\text{in}}^\nu(r') \theta(r - r') \right); \\
 W_{\ell m \omega}(\phi_{\text{in}}^\nu, \phi_{\text{up}}^\nu) &= r^2 \left(1 - \frac{2M}{r} \right) \left(\left(\frac{d}{dr} \phi_{\text{in}}^\nu(r) \right) \phi_{\text{up}}^\nu(r) - \left(\frac{d}{dr} \phi_{\text{up}}^\nu(r) \right) \phi_{\text{in}}^\nu(r) \right). \quad (\text{A} \cdot 1)
 \end{aligned}$$

For technical convenience, we express the homogeneous solutions ϕ_{in}^ν and ϕ_{up}^ν in terms of the solutions ϕ_c^ν and $\phi_c^{-\nu-1}$, which are expressed in terms of a series of Coulomb wave functions¹⁸⁾ as

$$\begin{aligned}\phi_{\text{in}}^\nu &= \alpha_\nu \phi_c^\nu + \beta_\nu \phi_c^{-\nu-1}, \\ \phi_{\text{up}}^\nu &= \gamma_\nu \phi_c^\nu + \delta_\nu \phi_c^{-\nu-1}.\end{aligned}\tag{A.2}$$

The properties of and relations among the coefficients $\{\alpha_\nu, \beta_\nu, \gamma_\nu, \delta_\nu\}$ are discussed extensively in Refs. 18) and 19). The function ϕ_c^ν is denoted by R_c^ν in Ref. 18). It may be noted that $\phi_c^\nu(r) \propto j_\ell(\omega r)$ in the limit $\epsilon \rightarrow 0$, where j_ℓ is the spherical Bessel function, and we may choose its phase so that ϕ_c^ν is real.

We then divide the Green function into time-symmetric and radiative parts as

$$g_{\ell m \omega}^{\text{full}}(r, r') = g_{\ell m \omega}^{(\text{sym})}(r, r') + g_{\ell m \omega}^{(\text{rad})}(r, r'),\tag{A.3}$$

where

$$\begin{aligned}g_{\ell m \omega}^{(\text{sym})}(r, r') &= \frac{1}{W_{\ell m \omega}(\phi_c^\nu, \phi_c^{-\nu-1})} \left[\theta(r' - r) \left\{ \phi_c^\nu(r) \phi_c^{-\nu-1}(r') \right. \right. \\ &\quad + \Re \left[\frac{\tilde{\beta}_\nu \tilde{\gamma}_\nu}{1 - \tilde{\beta}_\nu \tilde{\gamma}_\nu} \right] \left(\phi_c^\nu(r) \phi_c^{-\nu-1}(r') + \phi_c^{-\nu-1}(r) \phi_c^\nu(r') \right) \\ &\quad + \Re \left[\frac{\tilde{\gamma}_\nu}{1 - \tilde{\beta}_\nu \tilde{\gamma}_\nu} \right] \phi_c^\nu(r) \phi_c^\nu(r') + \Re \left[\frac{\tilde{\beta}_\nu}{1 - \tilde{\beta}_\nu \tilde{\gamma}_\nu} \right] \phi_c^{-\nu-1}(r) \phi_c^{-\nu-1}(r') \left. \right\} \\ &\quad + \theta(r - r') \left\{ \nu \leftrightarrow -\nu - 1, \quad \tilde{\beta}_\nu \leftrightarrow \tilde{\gamma}_\nu \right\} \Big],\end{aligned}\tag{A.4}$$

$$\begin{aligned}g_{\ell m \omega}^{(\text{rad})}(r, r') &= \frac{i}{W_{\ell m \omega}(\phi_c^\nu, \phi_c^{-\nu-1})} \left[\Im \left[\frac{\tilde{\beta}_\nu \tilde{\gamma}_\nu}{1 - \tilde{\beta}_\nu \tilde{\gamma}_\nu} \right] \left(\phi_c^\nu(r) \phi_c^{-\nu-1}(r') + \phi_c^{-\nu-1}(r) \phi_c^\nu(r') \right) \right. \\ &\quad + \Im \left[\frac{\tilde{\gamma}_\nu}{1 - \tilde{\beta}_\nu \tilde{\gamma}_\nu} \right] \phi_c^\nu(r) \phi_c^\nu(r') \\ &\quad \left. + \Im \left[\frac{\tilde{\beta}_\nu}{1 - \tilde{\beta}_\nu \tilde{\gamma}_\nu} \right] \phi_c^{-\nu-1}(r) \phi_c^{-\nu-1}(r') \right].\end{aligned}\tag{A.5}$$

In the above, we have assumed $\alpha_\nu \neq 0$ and $\delta_\nu \neq 0$ and introduced the coefficients $\{\tilde{\beta}_\nu, \tilde{\gamma}_\nu\} := \{\beta_\nu/\alpha_\nu, \gamma_\nu/\delta_\nu\}$. Using the result obtained in Ref. 18), we find that the coefficients $\{\tilde{\beta}_\nu, \tilde{\gamma}_\nu\}$ behave under the post-Newtonian expansion as

$$\begin{aligned}\Re[\tilde{\beta}_\nu] &= O(v^{6\ell+3}), & \Im[\tilde{\beta}_\nu] &= O(v^{6\ell}), \\ \Re[\tilde{\gamma}_\nu] &= O(v^6), & \Im[\tilde{\gamma}_\nu] &= (-1)^{\ell+1}.\end{aligned}\tag{A.6}$$

Although the analysis in Refs. 18) and 19) is very useful, it turns out that the post Newtonian expansion of $\phi_c^{-\nu-1}$ breaks down at $(\ell + 2)$ PN order. Thus, for our analysis to 3PN order, the modes $\ell = 0, 1$ require a separate treatment. So, we consider the three cases $g_{\ell m \omega}^{(\text{sym})}$ and $g_{\ell m \omega}^{(\text{rad})}$ for $\ell \geq 2$, and $g_{\ell m \omega}$ for $\ell = 0, 1$ separately.

A.1. Time-symmetric part for $\ell \geq 2$

To 3PN order, inspection of the post-Newtonian orders of $\{\tilde{\beta}_\nu, \tilde{\gamma}_\nu\}$ shows that the time-symmetric Green function for $\ell \geq 2$ can be approximated as

$$g_{\ell m \omega}^{(\text{sym})}(r, r') = \frac{1}{W_{\ell m \omega}(\phi_c^\nu, \phi_c^{-\nu-1})} \times \left[\theta(r' - r) \phi_c^\nu(r) \phi_c^{-\nu-1}(r') + \theta(r - r') \phi_c^{-\nu-1}(r) \phi_c^\nu(r') \right] + \dots \quad (\text{A.7})$$

The homogeneous solution ϕ_c^ν is expanded to $O(v^6)$ as

$$\begin{aligned} \phi_c^\nu(z) = (2z)^\nu & \left(1 - \frac{z^2}{2(2\ell+3)} - \frac{\ell\epsilon}{2z} + \frac{z^4}{8(2\ell+3)(2\ell+5)} \right. \\ & + \frac{(\ell^2 - 5\ell - 10)\epsilon z}{4(2\ell+3)(\ell+1)} + \frac{\ell(\ell-1)^2\epsilon^2}{4(2\ell-1)z^2} - \frac{z^6}{48(2\ell+3)(2\ell+5)(2\ell+7)} \\ & - \frac{(3\ell^3 - 27\ell^2 - 142\ell - 136)\epsilon z^3}{48(\ell+1)(\ell+2)(2\ell+3)(2\ell+5)} - \frac{(\ell^3 - 18\ell^2 + 17\ell - 4)\epsilon^2}{8(2\ell-1)^2} \\ & \left. - \frac{\ell(\ell-1)(\ell-2)^2\epsilon^3}{24(2\ell-1)z^3} + O(v^7) \right); \end{aligned} \quad (\text{A.8})$$

$$\nu = \ell - \frac{15\ell^2 + 15\ell - 11}{2(2\ell-1)(2\ell+1)(2\ell+3)}\epsilon^2, \quad (\text{A.9})$$

where $z = \omega r$, $\epsilon = 2M\omega$, $z \sim v$ and $\epsilon \sim v^3$ in the post Newtonian expansion. The solution $\phi_c^{-\nu-1}$ can be obtained by changing $\ell \rightarrow -\ell - 1$ for $\ell \neq 0, 1$. We point out that Eq. (A.8) exhibits breakdown of the post-Newtonian expansion of $\phi_c^{-\nu-1}$ for $\ell = 0$ and 1 at the second and third PN orders, respectively. The Wronskian of ϕ_c^ν and $\phi_c^{-\nu-1}$ becomes

$$\begin{aligned} \omega W_{\ell m \omega}(\phi_c^\nu, \phi_c^{-\nu-1}) &= -\frac{2\ell+1}{2} \\ &- \frac{32\ell^6 + 96\ell^5 - 176\ell^4 - 512\ell^3 + 78\ell^2 + 350\ell - 131}{8(2\ell-1)^2(2\ell+1)(2\ell+3)^2}\epsilon^2 + O(v^7). \end{aligned} \quad (\text{A.10})$$

It should be noted that ϕ_c^ν contains a fractional power of z , z^ν . Such a factor could be an obstacle to the power expansion. However, as seen from Eq. (A.4), since z^ν is associated with $z'^{-\nu-1}$ (or $z^{-\nu-1}$ with z'^ν), the fractional power appears always in the form $(z/z')^\nu = (r/r')^\nu$ [or $(z'/z)^\nu = (r'/r)^\nu$]. Therefore, it may be expanded in powers of $(r - r')$, and hence of r .

To determine up to which order of $(r - r')$ we have to expand for the 3PN calculation, we note the fact that the $O(\epsilon^2)[= O(v^6)]$ correction in ν gives a factor of

$$- \int d\omega \sum_m \frac{15\ell^2 + 15\ell - 11}{2(2\ell-1)(2\ell+1)(2\ell+3)}\epsilon^2$$

in front of the powers of $(r - r')$. Since $\omega = mv^t/v^\phi$ in the stationary phase approximation (which is exact for a circular orbit), the above term behaves as $\sim \ell$ after integrating over ω and taking the m -sum. The divergence after summing over ℓ will therefore be proportional to $(r - r')^{-2}$. Thus the expansion to $O((r - r')^3)$ is sufficient:

$$\begin{aligned} & \left(\frac{r}{r'} \right)^{-\frac{15\ell^2+15\ell-11}{2(2\ell-1)(2\ell+1)(2\ell+3)}\epsilon^2} \\ &= 1 - \frac{15\ell^2+15\ell-11}{2(2\ell-1)(2\ell+1)(2\ell+3)}\epsilon^2 \\ & \quad \times \left\{ \frac{r-r'}{r'} - \frac{1}{2} \left(\frac{r-r'}{r'} \right)^2 + \frac{1}{3} \left(\frac{r-r'}{r'} \right)^3 + O((r-r')^4) \right\}. \quad (\text{A}\cdot 11) \end{aligned}$$

The coefficients of the power expansion for the radial Green function $g_{\ell m \omega}^{(\text{sym})}$ are listed in Appendix B. We only note here that the leading term of the post-Newtonian expansion in $g_{\ell m \omega}^{(\text{sym})}$ behaves as $1/r$ or $1/r'$.

A.2. Radiative part for $\ell \geq 2$

To 3PN order, the post-Newtonian orders of $\{\tilde{\beta}_\nu, \tilde{\gamma}_\nu\}$ given in Eq. (A·6) imply that only the term proportional to $\phi_c^\nu(r)\phi_c^\nu(r')$ contributes to the radiative Green function $g_{\ell m \omega}^{(\text{rad})}$,

$$g_{\ell m \omega}^{(\text{rad})}(r, r') = \frac{\Im[\tilde{\gamma}_\nu] \phi_c^\nu(r)\phi_c^\nu(r')}{W_{\ell m \omega}(\phi_c^\nu, \phi_c^{-\nu-1})} + \dots \quad (\text{A}\cdot 12)$$

The leading order term is then

$$g_{\ell m \omega}^{(\text{rad})}(r, r') = i \frac{(-1)^\ell 2^{2\ell+1} \omega^{2\ell+1} r^\ell r'^\ell}{2\ell+1} \left(1 + O(v^2) \right), \quad (\text{A}\cdot 13)$$

which is $O(v^{2\ell+1})$ relative to the leading term in $g_{\ell m \omega}^{(\text{sym})}$. Hence, only the mode $\ell = 2$ is necessary to 3PN order.

It may be noted that in an expansion to a finite post-Newtonian order, only a finite number of ℓ contribute to the radiative Green function. Note also that because $g_{\ell m \omega}^{(\text{rad})}$ is purely imaginary, the radiative Green function does not contribute to the r -component of the reaction force in the present case of circular orbit around a black hole, whereas the t - and ϕ -components of the force are due to this part. This is in accordance with the fact that the radiative Green function describes the energy and angular momentum loss.

A.3. $\ell = 0$ and 1 modes

For the modes $\ell = 0, 1$, since the homogeneous function $\phi_c^{-\nu-1}$ cannot be used, we explicitly perform the post-Newtonian expansion of $\phi_{\text{in}}^{\nu(\ell)}$ and $\phi_{\text{up}}^{\nu(\ell)}$, as done in Refs. 30), 31) and 7). In the limit $\epsilon \rightarrow 0$, we have

$${}^{(0)}\phi_{\text{in}}^{\nu(0)}(z) = j_0(z) = 1 - \frac{z^2}{6} + \frac{z^4}{120} - \frac{z^6}{5040} + O(v^7), \quad (\text{A}\cdot 14)$$

$${}^{(0)}\phi_{\text{in}}^{\nu(1)}(z) = j_1(z) = \frac{z}{3} \left(1 - \frac{z^2}{10} + \frac{z^4}{280} - \frac{z^6}{15120} + O(v^7) \right), \quad (\text{A}\cdot 15)$$

$${}^{(0)}\phi_{\text{up}}^{\nu(0)}(z) = h_0^{(1)}(z) = \frac{-i}{z} \left(1 + iz - \frac{z^2}{2} - \frac{iz^3}{6} + \frac{z^4}{24} + \frac{iz^5}{120} - \frac{z^6}{720} + O(v^7) \right), \quad (\text{A}\cdot 16)$$

$${}^{(0)}\phi_{\text{up}}^{\nu(1)}(z) = h_1^{(1)}(z) = \frac{-i}{z^2} \left(1 + \frac{z^2}{2} + \frac{iz^3}{3} - \frac{z^4}{8} - \frac{iz^5}{30} + \frac{z^6}{144} + O(v^7) \right), \quad (\text{A}\cdot 17)$$

where j_n and $h_n^{(1)}$ are the spherical Bessel function and the spherical Hankel function of the first kind, respectively. Starting from the above, the expansion to 3PN order, that is, to $O(\epsilon^2)$, is easily carried out, yielding

$$\phi_{\text{in}}^{\nu(0)}(z) = 1 - \frac{z^2}{6} + \frac{z^4}{120} - \frac{z^6}{5040} + \epsilon \left(-\frac{5z}{6} + \frac{17z^3}{180} \right) - \frac{11}{6} \epsilon^2 \ln(z) + O(v^7), \quad (\text{A}\cdot 18)$$

$$\begin{aligned} \phi_{\text{in}}^{\nu(1)}(z) = \frac{z}{3} \left(1 - \frac{z^2}{10} + \frac{z^4}{280} - \frac{z^6}{15120} - \frac{1}{2} \frac{\epsilon}{z} - \frac{7z\epsilon}{20} + \frac{151z^3\epsilon}{5040} \right. \\ \left. - \frac{19}{30} \epsilon^2 \ln(z) + O(v^7) \right), \end{aligned} \quad (\text{A}\cdot 19)$$

$$\begin{aligned} \phi_{\text{up}}^{\nu(0)}(z) = \frac{-i}{z} \left(1 + iz - \frac{z^2}{2} - \frac{iz^3}{6} + \frac{z^4}{24} + \frac{iz^5}{120} - \frac{z^6}{720} + \frac{1}{2} \frac{\epsilon}{z} - \frac{34i\epsilon}{15} \right. \\ \left. - \frac{7\epsilon z}{24} + \frac{3i\epsilon z^2}{10} - \frac{z^3\epsilon}{12} + \frac{\epsilon^2}{3z^2} - \frac{17i\epsilon^2}{15z} + \frac{\epsilon^3}{4z^3} + \frac{11}{6} \epsilon^2 \ln(z) \right. \\ \left. - 2z\epsilon \ln(z) + \frac{\epsilon z^3}{3} \ln(z) + O(v^7) \right), \end{aligned} \quad (\text{A}\cdot 20)$$

$$\begin{aligned} \phi_{\text{up}}^{\nu(1)}(z) = \frac{-i}{z^2} \left(1 + \frac{z^2}{2} + \frac{iz^3}{3} - \frac{z^4}{8} - \frac{iz^5}{30} + \frac{z^6}{144} + \frac{\epsilon}{z} \right. \\ \left. - \frac{i\epsilon z^2}{6} + \epsilon z + \frac{9\epsilon^2}{10z^2} + \frac{4\epsilon^3}{5z^3} + \frac{19}{30} \epsilon^2 \ln(z) - \frac{2z^3\epsilon}{3} \ln(z) + O(v^7) \right). \end{aligned} \quad (\text{A}\cdot 21)$$

The Wronskians $W_{\ell m \omega} = W_{\ell m \omega}(\phi_{\text{in}}^{\nu(\ell)}, \phi_{\text{up}}^{\nu(\ell)})$ for $\ell = 0$ and 1 become

$$\omega W_{0m\omega} = \frac{i}{180} (-180 + 408i\epsilon + 935\epsilon^2) + O(v^7), \quad (\text{A}\cdot 22)$$

$$\omega W_{1m\omega} = \frac{i}{225} (-225 + 311\epsilon^2) + O(v^7). \quad (\text{A}\cdot 23)$$

From the above formulas, we find the full radial Green function for $\ell = 0$ and 1 as

$$\begin{aligned}
g_{0m\omega}(r, r') = & \frac{6 M^3 + 3 r_{<}^3 + 3 r_{<}^2 M + 4 r_{<} M^2}{3 r_{<}^4} + i\omega \\
& - \frac{1}{180 r_{<}^3} \left(300 M^2 r_{>} r_{<} + 90 r_{<}^4 + 30 r_{>}^2 r_{<} M - 3740 M^2 r_{<}^2 \right. \\
& \quad + 921 r_{<}^3 M + 1320 M^2 r_{<}^2 \ln(\omega r_{>}) + 300 M r_{>} r_{<}^2 \\
& \quad + 30 r_{>}^2 r_{<}^2 + 720 r_{<}^3 M \ln(\omega r_{<}) + 40 r_{>}^2 M^2 \\
& \quad \left. - 1320 M^2 \ln(\omega r_{<}) r_{<}^2 \right) \omega^2 \\
& - \frac{i}{6} \left(r_{<}^2 + r_{>}^2 + 10 M r_{<} + 10 M r_{>} \right) \omega^3 \\
& + \frac{1}{360 r_{<}^2} \left(30 r_{>}^2 r_{<}^3 + 307 r_{>}^2 r_{<}^2 M + 15 r_{<}^5 + 240 r_{>}^2 r_{<}^2 M \ln(\omega r_{<}) \right. \\
& \quad + 300 M r_{>} r_{<}^3 + 240 r_{<}^4 M \ln(\omega r_{<}) + 3 r_{>}^4 M \\
& \quad \left. + 68 M r_{>}^3 r_{<} + 212 r_{<}^4 M + 3 r_{>}^4 r_{<} \right) \omega^4 \\
& + \frac{1}{360} i \left(3 r_{<}^4 + 3 r_{>}^4 + 10 r_{>}^2 r_{<}^2 \right) \omega^5 \\
& - \frac{1}{5040} \frac{35 r_{>}^2 r_{<}^4 + 7 r_{<}^6 + 21 r_{>}^4 r_{<}^2 + r_{>}^6}{r_{<}} \omega^6, \tag{A.24} \\
g_{1m\omega}(r, r') = & \frac{1}{15 r_{<}^5} \left(-10 M r_{>} r_{<}^2 - 18 M^2 r_{>} r_{<} + 18 M^3 r_{<} - 5 r_{>} r_{<}^3 \right. \\
& \quad \left. + 5 r_{<}^3 M - 32 r_{>} M^3 + 10 M^2 r_{<}^2 \right) \\
& - \frac{1}{1350 r_{<}^4} \left(630 r_{>}^2 r_{<} M^2 - 225 r_{>} r_{<}^4 + 1140 r_{>} M^2 r_{<}^2 \ln(\omega r_{>}) \right. \\
& \quad + 225 r_{<}^4 M + 900 M^2 r_{<}^3 + 162 r_{>}^3 M^2 \\
& \quad - 2488 M^2 r_{>} r_{<}^2 + 45 r_{>}^3 r_{<}^2 - 1140 r_{>} M^2 \ln(\omega r_{<}) r_{<}^2 \\
& \quad \left. + 90 M r_{>}^3 r_{<} + 315 r_{>}^2 r_{<}^2 M - 900 M r_{>} r_{<}^3 \right) \omega^2 \\
& - \frac{i}{9} (-r_{>} r_{<} + M r_{<} + M r_{>}) \omega^3 \\
& + \frac{1}{7560 r_{<}^3} \left(315 M r_{<}^5 - 882 M r_{>}^2 r_{<}^3 - 126 r_{>}^3 r_{<}^3 + 9 r_{>}^5 r_{<} \right. \\
& \quad + 18 r_{>}^5 M - 3360 r_{>} M \ln(\omega r_{<}) r_{<}^4 + 151 M r_{>}^4 r_{<} \\
& \quad \left. - 315 r_{>} r_{<}^5 - 504 r_{>}^3 r_{<}^2 M \right) \omega^4 \\
& - \frac{1}{90} i r_{>} r_{<} \left(r_{<}^2 + r_{>}^2 \right) \omega^5
\end{aligned}$$

$$+ \frac{1}{45360} \frac{r_{>} (105 r_{<}^6 + 189 r_{>}^2 r_{<}^4 + 27 r_{>}^4 r_{<}^2 - r_{>}^6)}{r_{<}^2} \omega^6, \quad (\text{A} \cdot 25)$$

where $r_{>} = \max\{r, r'\}$ and $r_{<} = \min\{r, r'\}$.

Appendix B

Power Expansion Coefficients

In this appendix, we consider the power expansion of the time-symmetric part of the radial Green function $g_{\ell m \omega}^{(\text{sym})}$. It is expanded as

$$g_{\ell m \omega}^{(\text{sym})}(r, r'; r_0) = \sum_n \left(\theta(r' - r) \frac{r'^{n+\ell}}{r'^{n+\ell+1}} g_{\ell m \omega}^{\text{in}(n)}(r', r_0) + \theta(r - r') \frac{r'^{n+\ell}}{r'^{n+\ell+1}} g_{\ell m \omega}^{\text{out}(n)}(r', r_0) \right), \quad (\text{B} \cdot 1)$$

where the coefficients necessary for the 3PN order calculation are

$$g_{\ell m \omega}^{\text{in}(-3)}(r', r_0) = \frac{(\ell^3 - 5\ell^2 + 8\ell - 4)\ell}{3(2\ell - 1)(2\ell + 1)} \frac{M^3}{r'^3}, \quad (\text{B} \cdot 2)$$

$$g_{\ell m \omega}^{\text{in}(-2)}(r', r_0) = -\frac{(\ell - 1)^2 \ell}{(2\ell - 1)(2\ell + 1)} \frac{M^2}{r'^2} - \frac{(\ell - 1)^2 \ell}{2(2\ell - 1)^2(2\ell + 1)} (\omega M)^2 - \frac{\ell(\ell^3 - \ell^2 - \ell + 1)}{(2\ell - 1)(2\ell + 1)} \frac{M^3}{r'^3}, \quad (\text{B} \cdot 3)$$

$$g_{\ell m \omega}^{\text{in}(-1)}(r', r_0) = \frac{\ell}{2\ell + 1} \frac{M}{r'} + \frac{\ell}{2(2\ell - 1)(2\ell + 1)} (\omega r') (\omega M) + \frac{\ell(\ell + 1)}{2\ell + 1} \frac{M^2}{r'^2} + \frac{\ell}{8(2\ell - 3)(2\ell - 1)(2\ell + 1)} (\omega r')^3 (\omega M) + \frac{\ell^2 + 7\ell - 4}{2(2\ell - 1)(2\ell + 1)} (\omega M)^2 + \frac{\ell(\ell^3 + 5\ell^2 + 8\ell + 4)}{(2\ell + 1)(2\ell + 3)} \frac{M^3}{r'^3}, \quad (\text{B} \cdot 4)$$

$$g_{\ell m \omega}^{\text{in}(+0)}(r', r_0) = -\frac{1}{2\ell + 1} - \frac{1}{2(2\ell - 1)(2\ell + 1)} (\omega r')^2 - \frac{\ell + 1}{2\ell + 1} \frac{M}{r'} - \frac{1}{8(2\ell - 3)(2\ell - 1)(2\ell + 1)} (\omega r')^4 - \frac{\ell^2 + 7\ell - 4}{2\ell(2\ell - 1)(2\ell + 1)} (\omega r') (\omega M) - \frac{(\ell + 1)(\ell + 2)^2}{(2\ell + 1)(2\ell + 3)} \frac{M^2}{r'^2} - \frac{1}{48(2\ell - 5)(2\ell - 3)(2\ell - 1)(2\ell + 1)} (\omega r')^6 - \frac{3\ell^3 + 36\ell^2 - 79\ell + 24}{24(\ell - 1)\ell(2\ell - 3)(2\ell - 1)(2\ell + 1)} (\omega r')^3 (\omega M) - (1488\ell^6 + 7104\ell^5 + 10608\ell^4 + 1940\ell^3 - 6825\ell^2 - 2321\ell + 1740)/(6(2\ell - 1)^2(2\ell + 1)^3(2\ell + 3)^2) (\omega M)^2$$

$$\begin{aligned}
& + \frac{2(15\ell^2 + 15\ell - 11)}{(2\ell - 1)(2\ell + 1)^2(2\ell + 3)} (\omega M)^2 \ln(r_0/r') \\
& - \frac{\ell^4 + 9\ell^3 + 29\ell^2 + 39\ell + 18}{3(2\ell + 1)(2\ell + 3)} \frac{M^3}{r'^3}, \tag{B.5}
\end{aligned}$$

$$\begin{aligned}
g_{\ell m \omega}^{\text{in}(+1)}(r', r_0) = & - \frac{\ell^2 - 5\ell - 10}{2(\ell + 1)(2\ell + 1)(2\ell + 3)} (\omega r') (\omega M) \\
& - \frac{\ell^2 - 5\ell - 10}{4(\ell + 1)(2\ell - 1)(2\ell + 1)(2\ell + 3)} (\omega r')^3 (\omega M) \\
& + \frac{6(15\ell^2 + 15\ell - 11)}{(2\ell - 1)(2\ell + 1)^2(2\ell + 3)} \frac{r'}{r_0} (\omega M)^2 \\
& - \frac{\ell^2 - 5\ell - 10}{2(2\ell + 1)(2\ell + 3)} (\omega M)^2, \tag{B.6}
\end{aligned}$$

$$\begin{aligned}
g_{\ell m \omega}^{\text{in}(+2)}(r', r_0) = & \frac{1}{2(2\ell + 1)(2\ell + 3)} (\omega r')^2 + \frac{1}{4(2\ell - 1)(2\ell + 1)(2\ell + 3)} (\omega r')^4 \\
& + \frac{\ell + 1}{2(2\ell + 1)(2\ell + 3)} (\omega r') (\omega M) \\
& + \frac{1}{16(2\ell - 3)(2\ell - 1)(2\ell + 1)(2\ell + 3)} (\omega r')^6 \\
& + \frac{\ell^2 + 7\ell - 4}{4\ell(2\ell - 1)(2\ell + 1)(2\ell + 3)} (\omega r')^3 (\omega M) \\
& + \frac{3(15\ell^2 + 15\ell - 11)}{(2\ell - 1)(2\ell + 1)^2(2\ell + 3)} \frac{r'^2}{r_0^2} (\omega M)^2 \\
& - \frac{(\ell + 1)(\ell + 2)^2}{2(2\ell + 1)(2\ell + 3)^2} (\omega M)^2, \tag{B.7}
\end{aligned}$$

$$\begin{aligned}
g_{\ell m \omega}^{\text{in}(+3)}(r', r_0) = & \frac{3\ell^3 - 27\ell^2 - 142\ell - 136}{24(2\ell^3 + 11\ell^2 + 19\ell + 10)(2\ell + 3)(2\ell + 1)} (\omega r')^3 (\omega M) \\
& + \frac{2(15\ell^2 + 15\ell - 11)}{3(2\ell - 1)(2\ell + 1)^2(2\ell + 3)} \left(\frac{r'}{r_0} \right)^3 (\omega M)^2, \tag{B.8}
\end{aligned}$$

$$\begin{aligned}
g_{\ell m \omega}^{\text{in}(+4)}(r', r_0) = & - \frac{1}{8(2\ell + 1)(2\ell + 3)(2\ell + 5)} (\omega r')^4 \\
& - \frac{1}{16(2\ell - 1)(2\ell + 1)(2\ell + 3)(2\ell + 5)} (\omega r')^6 \\
& - \frac{\ell + 1}{8(2\ell + 1)(2\ell + 3)(2\ell + 5)} (\omega r')^3 (\omega M), \tag{B.9}
\end{aligned}$$

$$g_{\ell m \omega}^{\text{in}(+5)}(r', r_0) = 0, \tag{B.10}$$

$$g_{\ell m \omega}^{\text{in}(+6)}(r', r_0) = \frac{1}{48(2\ell + 7)(2\ell + 1)(2\ell + 3)(2\ell + 5)} (\omega r')^6, \tag{B.11}$$

$$g_{\ell m \omega}^{\text{out}(+3)}(r', r_0) = - \frac{\ell^4 + 9\ell^3 + 29\ell^2 + 39\ell + 18}{3(2\ell + 1)(2\ell + 3)} \frac{M^3}{r'^3}, \tag{B.12}$$

$$g_{\ell m \omega}^{\text{out}(+2)}(r', r_0) = -\frac{(\ell+1)(\ell+2)^2}{(2\ell+1)(2\ell+3)} \frac{M^2}{r'^2} + \frac{(\ell+1)(\ell+2)^2}{2(2\ell+1)(2\ell+3)^2} (\omega M)^2 \\ + \frac{\ell(\ell+1)(\ell+2)^2}{(2\ell+1)(2\ell+3)} \frac{M^3}{r'^3}, \quad (\text{B}\cdot 13)$$

$$g_{\ell m \omega}^{\text{out}(+1)}(r', r_0) = -\frac{\ell+1}{2\ell+1} \frac{M}{r'} + \frac{\ell+1}{2(2\ell+1)(2\ell+3)} (\omega r') (\omega M) \\ + \frac{\ell(\ell+1)}{2\ell+1} \frac{M^2}{r'^2} - \frac{\ell+1}{8(2\ell+1)(2\ell+3)(2\ell+5)} (\omega r')^3 (\omega M) \\ - \frac{\ell^2 - 5\ell - 10}{2(2\ell+1)(2\ell+3)} (\omega M)^2 - \frac{(\ell-1)^2 \ell (\ell+1)}{(2\ell-1)(2\ell+1)} \frac{M^3}{r'^3}, \quad (\text{B}\cdot 14)$$

$$g_{\ell m \omega}^{\text{out}(+0)}(r', r_0) = -\frac{1}{2\ell+1} + \frac{1}{2(2\ell+1)(2\ell+3)} (\omega r')^2 + \frac{\ell}{2\ell+1} \frac{M}{r'} \\ - \frac{1}{8(2\ell+1)(2\ell+3)(2\ell+5)} (\omega r')^4 \\ - \frac{\ell^2 - 5\ell - 10}{2(\ell+1)(2\ell+1)(2\ell+3)} (\omega r') (\omega M) - \frac{\ell(\ell-1)^2}{(2\ell-1)(2\ell+1)} \frac{M^2}{r'^2} \\ + \frac{1}{48(2\ell+1)(2\ell+3)(2\ell+5)(2\ell+7)} (\omega r')^6 \\ + \frac{3\ell^3 - 27\ell^2 - 142\ell - 136}{24(\ell+1)(\ell+2)(2\ell+1)(2\ell+3)(2\ell+5)} (\omega r')^3 (\omega M) \\ - (1488\ell^6 + 1824\ell^5 - 2592\ell^4 - 788\ell^3 + 2283\ell^2 \\ - 1309\ell + 288) / (6(2\ell-1)^2(2\ell+1)^3(2\ell+3)^2) (\omega M)^2 \\ + \frac{2(15\ell^2 + 15\ell - 11)}{(2\ell-1)(2\ell+1)^2(2\ell+3)} (\omega M)^2 \ln(r'/r_0) \\ + \frac{(\ell^3 - 5\ell^2 + 8\ell - 4)\ell}{3(2\ell+1)(2\ell-1)} \frac{M^3}{r'^3}, \quad (\text{B}\cdot 15)$$

$$g_{\ell m \omega}^{\text{out}(-1)}(r', r_0) = -\frac{\ell^2 + 7\ell - 4}{2\ell(2\ell-1)(2\ell+1)} (\omega r') (\omega M) \\ + \frac{\ell^2 + 7\ell - 4}{4(2\ell-1)(2\ell+1)(2\ell+3)} (\omega r')^3 (\omega M) \\ - \frac{6(15\ell^2 + 15\ell - 11)}{(2\ell-1)(2\ell+1)^2(2\ell+3)} \frac{r'}{r_0} (\omega M)^2 \\ + \frac{\ell^2 + 7\ell - 4}{2(2\ell-1)(2\ell+1)} (\omega M)^2, \quad (\text{B}\cdot 16)$$

$$g_{\ell m \omega}^{\text{out}(-2)}(r', r_0) = -\frac{1}{2(2\ell-1)(2\ell+1)} (\omega r')^2 + \frac{1}{4(2\ell-1)(2\ell+1)(2\ell+3)} (\omega r')^4 \\ + \frac{\ell}{2(2\ell-1)(2\ell+1)} (\omega r') (\omega M) \\ - \frac{1}{16(2\ell-1)(2\ell+1)(2\ell+3)(2\ell+5)} (\omega r')^6$$

$$\begin{aligned}
& -\frac{\ell^2 - 5\ell - 10}{4(\ell + 1)(2\ell - 1)(2\ell + 1)(2\ell + 3)}(\omega r')^3(\omega M) \\
& + \frac{3(15\ell^2 + 15\ell - 11)}{(2\ell - 1)(2\ell + 1)^2(2\ell + 3)}\left(\frac{r'}{r_0}\right)^2(\omega M)^2 \\
& + \frac{(\ell - 1)^2\ell}{2(2\ell - 1)^2(2\ell + 1)}(\omega M)^2, \tag{B.17}
\end{aligned}$$

$$\begin{aligned}
g_{\ell m \omega}^{\text{out}(-3)}(r', r_0) &= -\frac{3\ell^3 + 36\ell^2 - 79\ell + 24}{24(\ell - 1)\ell(2\ell - 3)(2\ell - 1)(2\ell + 1)}(\omega r')^3(\omega M) \\
& - \frac{2(15\ell^2 + 15\ell - 11)}{3(2\ell - 1)(2\ell + 1)^2(2\ell + 3)}\left(\frac{r'}{r_0}\right)^3(\omega M)^2, \tag{B.18}
\end{aligned}$$

$$\begin{aligned}
g_{\ell m \omega}^{\text{out}(-4)}(r', r_0) &= -\frac{1}{8(2\ell - 3)(2\ell - 1)(2\ell + 1)}(\omega r')^4 \\
& + \frac{1}{16(2\ell - 3)(2\ell - 1)(2\ell + 1)(2\ell + 3)}(\omega r')^6 \\
& + \frac{\ell}{8(2\ell - 3)(2\ell - 1)(2\ell + 1)}(\omega r')^3(\omega M), \tag{B.19}
\end{aligned}$$

$$g_{\ell m \omega}^{\text{out}(-5)}(r', r_0) = 0, \tag{B.20}$$

$$g_{\ell m \omega}^{\text{out}(-6)}(r', r_0) = -\frac{1}{48(2\ell - 5)(2\ell + 1)(2\ell - 1)(2\ell - 3)}(\omega r')^6. \tag{B.21}$$

Appendix C

— Generating Function of m -Sum —

In this appendix, we describe a method to sum over the m -modes of spherical harmonics for arbitrary ℓ that appears in Eq. (3.19). Specifically, the m -sum we need to evaluate takes the form

$$\sum_{m=-\ell}^{\ell} m^N |Y_{\ell m}(\pi/2, 0)|^2, \tag{C.1}$$

where N is a non-negative integer. To perform the above summation, we introduce the generating function

$$\Gamma_{\ell}(z) = \sum_{m=-\ell}^{\ell} e^{mz} |Y_{\ell m}(\pi/2, 0)|^2. \tag{C.2}$$

Then the sum (C.1) may be evaluated as $\lim_{z \rightarrow 0} \partial_z^N \Gamma_{\ell}(z)$.

First, note that $Y_{\ell m}(\pi/2, 0)$ is non-zero only when $\ell - m$ is an even integer. Hence, putting $2n = \ell - m$, we have

$$\Gamma_{\ell}(z) = \sum_{n=0}^{\ell} e^{(l-2n)z} \frac{2\ell + 1}{4\pi} \frac{(2n)!}{(2\ell - 2n)!} \left(\frac{(2\ell - 2n - 1)!!}{(2n)!!} \right)^2$$

$$\begin{aligned}
&= \frac{2\ell+1}{4\pi} \frac{e^{\ell z}}{4^\ell} \sum_{n=0}^{\ell} \frac{\Gamma(2n+1)\Gamma(2\ell-2n+1)}{\Gamma(n+1)\Gamma(\ell-n+1)^2} \frac{e^{-2nz}}{n!} \\
&= \frac{2\ell+1}{4\pi^2} \frac{\Gamma(\ell+1/2)\Gamma(-\ell+1/2)}{\Gamma(\ell+1)\Gamma(-\ell)} e^{\ell z} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)\Gamma(n-\ell)}{\Gamma(n-\ell+1/2)} \frac{e^{-2nz}}{n!} \\
&= \frac{2\ell+1}{4\pi^2} \frac{\Gamma(\ell+1/2)\Gamma(1/2)}{\Gamma(\ell+1)} e^{\ell z} {}_2F_1\left(\frac{1}{2}, -\ell; -\ell + \frac{1}{2}; e^{-2z}\right), \quad (\text{C.3})
\end{aligned}$$

where ${}_2F_1$ is a hypergeometric function, and we have temporarily performed the analytic continuation of ℓ to a complex number from the second line to the third line.

Using the relation between hypergeometric functions

$${}_2F_1\left(\frac{1}{2}, -\ell; -\ell + \frac{1}{2}; x\right) = (-1)^\ell \frac{\Gamma(\ell+1)\Gamma(-\ell+1/2)}{\Gamma(1/2)} {}_2F_1\left(\frac{1}{2}, -\ell; 1; 1-x\right), \quad (\text{C.4})$$

for a non-negative integer ℓ , we have

$$\Gamma_\ell(z) = \frac{2\ell+1}{4\pi} e^{\ell z} {}_2F_1\left(\frac{1}{2}, -\ell; 1; 1-e^{-2z}\right). \quad (\text{C.5})$$

This can be easily expanded to arbitrary order in z . For example, to $O(z^6)$ we have

$$\begin{aligned}
\Gamma_\ell(z) = \frac{2\ell+1}{4\pi} \left\{ 1 + \left(\frac{\ell(\ell+1)}{2}\right) \frac{1}{2} z^2 + \left(\frac{\ell(\ell+1)(3\ell^2+3\ell-2)}{8}\right) \frac{1}{4!} z^4 \right. \\
\left. + \left(\frac{\ell(\ell+1)(5\ell^4+10\ell^3-5\ell^2-10\ell+8)}{16}\right) \frac{1}{6!} z^6 + O(z^8) \right\}. \quad (\text{C.6})
\end{aligned}$$

References

- 1) Y. Mino, H. Nakano and M. Sasaki, *Covariant Self-force Regularization of a Particle Orbiting a Schwarzschild Black Hole*, UTAP-386, OUTAP-157 (2001).
- 2) LIGO web page: <http://www.ligo.caltech.edu/>
- 3) VIRGO web page: <http://www.virgo.infn.it/>
- 4) GEO600 web page: <http://www.geo600.uni-hannover.de/>
- 5) TAMA300 web page: <http://tamago.mtk.nao.ac.jp/>
- 6) LISA web page: <http://lisa.jpl.nasa.gov/>
- 7) Y. Mino, M. Sasaki, M. Shibata, H. Tagoshi and T. Tanaka, Prog. Theor. Phys. Suppl. No. 128 (1997), 1.
- 8) B. Carter, Phys. Rev. **174** (1968), 1559; Commun. Math. Phys. **10** (1968), 280.
- 9) Y. Mino, M. Sasaki and T. Tanaka, Phys. Rev. D **55** (1997), 3457; Prog. Theor. Phys. Suppl. No. 128 (1997), 373.
- 10) T. C. Quinn and R. M. Wald, Phys. Rev. D **56** (1997), 3381; **60** (1999), 064009.
- 11) J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations* (Yale University Press, New Haven, 1923).
- 12) B. S. DeWitt and R. W. Brehme, Ann. of Phys. **9** (1960), 220.
- 13) P. L. Chrzanowski, Phys. Rev. D **11** (1975), 2042.
- 14) S. A. Teukolsky, Astrophys. J. **185** (1973), 635.

- 15) Y. Mino, *Prog. Theor. Phys.* **99** (1998), 79.
- 16) Y. Mino and H. Nakano, *Prog. Theor. Phys.* **100** (1998), 507.
- 17) H. Nakano and M. Sasaki, *Prog. Theor. Phys.* **205** (2001), 197, gr-qc/0010036.
- 18) S. Mano, H. Suzuki and E. Takasugi, *Prog. Theor. Phys.* **95** (1996), 1079; **96** (1996), 549.
- 19) S. Mano and E. Takasugi, *Prog. Theor. Phys.* **97** (1997), 213.
- 20) L. Barack and A. Ori, *Phys. Rev. D* **61** (2000), 061502.
- 21) L. M. Burko, *Phys. Rev. Lett.* **84** (2000), 4529; *Class. Quant. Grav.* **17** (2000), 227.
- 22) L. Barack and L. M. Burko, *Phys. Rev. D* **62** (2000), 084040.
- 23) L. Barack, *Phys. Rev. D* **62** (2000), 084027.
- 24) C. O. Lousto, *Phys. Rev. Lett.* **84** (2000), 5251.
- 25) L. M. Burko, Y. T. Liu and Y. Soen, *Phys. Rev. D* **63** (2001), 024015.
- 26) S. Detweiler, *Phys. Rev. Lett.* **86** (2001), 1931, gr-qc/0011039.
- 27) T. C. Quinn, *Phys. Rev. D* **62** (2000), 064029.
- 28) P. L. Chrzanowski and C. W. Misner, *Phys. Rev. D* **10** (1974), 1701.
- 29) D. V. Gal'tsov, *J. of Phys. A* **15** (1982), 3737.
- 30) M. Sasaki, *Prog. Theor. Phys.* **92** (1994), 17.
- 31) E. Poisson and M. Sasaki, *Phys. Rev. D* **51** (1995), 5753.